SIAM J. CONTROL
Vol. 8, No. 4, November 1970

SOME SUFFICIENT CONDITIONS FOR THE GLOBEL AND LOCAL CONTROLLABILITY OF NONLINEAR TIME-VARYING SYSTEM*

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Summary. Sufficient conditions are derived for global and local controllability of nonlinear time-varying systems with control appearing linearly. It is shown that the controllability of $\dot{x} = A(t, x)x + B(t, x)u$ can be related to the controllability of the linear system $\dot{x} = A(t, z)x + B(t, z)u$, where z belongs to a certain set of continuous vector functions. This result is then used to specify a class of nonlinear systems which are globally controllable.

1. Introduction. This paper deals with the controllability of nonlinear time-varying systems with control appearing linearly. This problem has been studied by Hermes [1], who showed that a nonlinear system is locally controllable, if the associated Pfaffian system is not integrable. Markus and Lee [2] and Kalman [3] obtained conditions for local controllability of a more general class of systems than will be considered here, by showing that the nonlinear system is locally controllable if the linearized system is completely controllable.

The results obtained in this paper provide sufficient conditions for *complete* and *total controllability* as defined by Kreindler and Sarachik [4] for linear systems. A distinction is made between *global controllability* (i.e., the system is controllable in the whole of the state space R^n) and *local controllability* (i.e., the system is controllable only in some domain of R^n). It is shown (Theorem 1) that the system $\dot{x} = A(t, x)x + B(t, x)u$ is globally completely (totally) controllable, if the linear system $\dot{x} = A(t, z)x + B(t, z)u$ is completely (totally) controllable for all functions $z \in C_n[t_0, t_f]$. If the linear system is controllable only for z in some bounded family $\mathscr{C}$, then a criterion for local controllability results (Theorem 2). Theorem 3 shows how the controllability matrix $Q(t, z)$ introduced by Silverman and Meadows [5] can be used to test whether the linear system is controllable for all functions z belonging to $C_n[t_0, t_f]$ or $\mathscr{C}$, thereby giving a simple computable criterion for the global or local controllability of single input nonlinear systems. This criterion is then used to specify a class of nonlinear systems which are globally controllable (Theorem 4).

In deriving these results, the problem is transformed into one of showing the existence of a fixed point for a mapping $x = P(z)$, which is solved by using Schauder's fixed-point theorem. The existence of a fixed point requires that the determinant of Kalman's controllability matrix [1] of the linear system, here denoted by $G(t_0, t_f; z)$ with initial time t_0 and final time t_f, have a positive lower bound relative to z in $C_n[t_0, t_f]$ or $\mathscr{C}$.

2. Preliminaries. Consider the nonlinear time-varying system with linear control represented by the equation

$$dx/dt = A(t, x)x + B(t, x)u, \qquad t_0 \leqq t < \infty, \tag{1}$$

* Received by the editors November 6, 1969, and in revised form March 3, 1970. This work was supported by the National Research Council of Canada under Grant A 4396.

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where the state x is an n-vector, the control input u an m-vector, A is an $n \times n$ and B an $n \times m$ matrix. Assume that the elements $a_{ik}(t, x)$ of A ($i, k = 1, 2, 3, \cdots, n$) and the elements $b_{il}(t, x)$ of B ($i = 1, 2, 3, \cdots, n$, $l = 1, 2, 3, \cdots, m$) are continuous functions of x for fixed t and piecewise continuous functions of t for fixed x and fulfill the following conditions:

$$(2) \qquad |a_{ik}(t, x)| \leqq M, \quad |b_{il}(t, x)| \leqq N \quad \text{for all } x \in R^n, \quad t \in [t_0, t_f],$$

where M and N are positive real constants.

The following definitions are due to Kreindler and Sarachik [4].

DEFINITION 1. The system (1) is said to be *completely state controllable* at t_0 in the domain of controllability $D \subset R^n$, if each initial state $x(t_0)$ in D can be transferred to any final state x_f in D in some finite time $t_f(x_f) \geqq t_0$. (If D is the whole state space R^n, the controllability is said to be *global*. If D is not the whole of R^n, the controllability is said to be *local*.)

DEFINITION 2. The system given by (1) is said to be *totally state controllable* in the domain of controllability D, if it is completely state controllable in D on every interval $[t_0, t_f]$, $t_f > t_0$. (If D is the whole of R^n, the controllability is said to be *global*, otherwise it is said to be *local*.)

To derive sufficient conditions for the controllability of system (1) consider first the simpler system

$$(3) \qquad dx/dt = A(t, z)x + B(t, z)u,$$

where the argument x of A and B has been replaced by a specified function $z \in C_n[t_0, t_f]$, the Banach space of continuous R^n-valued functions on $[t_0, t_f]$. For each fixed $z \in C_n[t_0, t_f]$, system (3) is linear; and with $x(t_0) = x_0$, the solution is given by

$$(4) \qquad x(t) = \phi(t, t_0; z)x_0 + \int_{t_0}^{t} \phi(t, \tau; z)B(\tau, z)u(\tau)\, d\tau.$$

In (4), $\phi(t, t_0; z)$ is the state transition matrix of the system

$$(5) \qquad dx/dt = A(t, z)x$$

and is determined by

$$(6) \qquad \frac{d}{dt}\phi(t, t_0; z) = A(t, z)\phi(t, t_0; z), \qquad \phi(t_0, t_0; z) = I,$$

where I is the identity matrix. Define

$$(7) \qquad H(t_0, \tau; z) = \phi(t_0, \tau; z)B(\tau, z),$$

$$(8) \qquad G(t_0, t; z) = \int_{t_0}^{t} H(t_0, \tau; z)H'(t_0, \tau; z)\, d\tau.$$

The prime indicates the matrix transpose.

Necessary and sufficient conditions for system (3) to be controllable are summarized by the following lemmas (Kreindler and Sarachik [4]).

LEMMA 1. *System* (3) *is completely state controllable at* t_0 *if and only if there exists a finite time* $t_f > t_0$ *such that the rows of the matrix* $H(t_0, \tau; z)$ *are linearly independent functions of* τ *on* $[t_0, t_f]$.

where the state x is an n-vector, the control input u an m-vector, A is an $n \times n$ and B an $n \times m$ matrix. Assume that the elements $a_{ik}(t, x)$ of A $(i, k = 1, 2, 3, \cdots, n)$ and the elements $b_{il}(t, x)$ of B $(i = 1, 2, 3, \cdots, n, l = 1, 2, 3, \cdots, m)$ are continuous functions of x for fixed t and piecewise continuous functions of t for fixed x and fulfill the following conditions:

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To derive sufficient conditions for the controllability of system (1) consider first the simpler system

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In (4), $\phi(t, t_0; z)$ is the state transition matrix of the system

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LEMMA 1. *System* (3) *is completely state controllable at* t_0 *if and only if there exists a finite time* $t_f > t_0$ *such that the rows of the matrix* $H(t_0, \tau; z)$ *are linearly independent functions of* τ *on* $[t_0, t_f]$.

LEMMA 2. *System* (3) *is totally state controllable if and only if for all* t_0 *and for all* $t_f > t_0$ *the rows of matrix* $H(t_0, \tau; z)$ *are linearly independent functions of* τ *on* $[t_0, t_f]$.

3. Derivation of results—global controllability. Assume that system (3) is either completely or totally controllable for all $z \in C_n[t_0, t_f]$. For complete controllability, by Lemma 1, the rows of $H(t_0, \tau; z)$ are linearly independent functions of τ on some $[t_0, t_f]$. This implies that the matrix $G(t_0, t_f; z)$ defined by (8) (the Gramian matrix for the set of m-dimensional vector functions H on the interval $[t_0, t_f]$) is positive definite for $t = t_f$. Total controllability, by Lemma 2, then implies that the Gramian matrix $G(t_0, t_f; z)$ is positive definite for all t_0 and all $t_f > t_0$. In either case a control u always exists such that the system (3) can be transferred from any $x_0 \in R^n$ to any $x_f \in R^n$ in a finite time. Consider the control

$$u(t_0, t, t_f; z) = H'(t_0, t; z)G(t_0, t_f; z)^{-1}[\phi(t_f, t_0; z)^{-1}x_f - x_0]. \tag{9}$$

Using (7) and (8) and inserting (9) into (4), we obtain from (3):

$$x(t) = \phi(t, t_0; z)\{x_0 + G(t_0, t; z)G(t_0, t_f; z)^{-1}[\phi(t_f, t_0; z)^{-1}x_f - x_0]\}; \tag{10}$$

and it is easily verified that

$$x(t_0) = x_0, \qquad x(t_f) = x_f.$$

Clearly $u(t_0, t, t_f; z)$ as defined by (9) will transfer the system from x_0 to x_f for all $z \in C_n[t_0, t_f]$. In the following discussion it will be convenient to view the right side of (10) as an operator $P(z)(t)$, i.e.,

$$P(z)(t) = \phi(t, t_0; z)\{x_0 + G(t_0, t; z)G(t_0, t_f; z)^{-1}[\phi(t_f, t_0; z)^{-1}x_f - x_0]\} \tag{11}$$

so that (10) can be written in the form

$$x = P(z). \tag{12}$$

The following theorem now gives conditions under which the nonlinear system (1) is globally controllable.

THEOREM 1 (Global controllability). *The system*

$$dx/dt = A(t, x)x + B(t, x)u$$

is globally (a) *completely state controllable at* t_0 *or* (b) *totally state controllable, if the following three conditions all hold:*

(A) *The elements* $a_{ik}(t, x)$ *of* A $(i, k = 1, 2, \cdots, n)$ *and* $b_{il}(t, x)$ *of* B $(l = 1, 2, \cdots, m,\ i = 1, 2, \cdots, n)$ *are piecewise continuous functions of* t *and continuous functions of* x.

(B) $|a_{ik}(t, x)| \leqq M$, $|b_{il}(t, x)| \leqq N$, *for all* $x \in R^n$, $t \in [t_0, t_f]$, *where* M *and* N *are positive real constants.*

(C) *There exists a constant* $c > 0$ *such that*

$$\inf_{z \in C_n[t_0, t_f]} \det G(t_0, t_f; z) \geqq c$$

(a) *for some* $t_f > t_0$, *in the case of complete state controllability at* t_0,

(b) *for all* t_0 *and for all* $t_f > t_0$, *in the case of total state controllability.*

The proof of the theorem will be based on the following lemma.

LEMMA 3. *If conditions* (A), (B), (C) *of Theorem* 1 *are satisfied, then for every pair* $x_0, x_f \in R^n$, *the operator* P *defined by* (11) *has a fixed point in* $C_n[t_0, t_f]$.

Proof of Lemma 3. Define $|z(t)| = \sum_{i=1}^n |z_i(t)|$ and let the norm in $C_n[t_0, t_f]$ be

$$\|z\| = \max \{|z(t)| : t_0 \leqq t \leqq t_f\}. \tag{13}$$

Consider the closed and convex subset of $C_n[t_0, t_f]$:

$$\psi \equiv \{z | z \in C_n[t_0, t_f], \|z\| \leqq K\}, \tag{14}$$

where the constant K is defined by

$$K = \{(1 + C)|x_0| + C|x_f|e^{nM(t_f - t_0)}\}e^{nM(t_f - t_0)} \tag{15}$$

with

$$C = \sup_{z \in C_n[t_0, t_f]} \|G(t_0, t_f; z)^{-1}\| \, nmN^2(t_f - t_0)e^{2nM(t_f - t_0)} \tag{16}$$

where

$$\|G(t_0, t_f; z)^{-1}\| = \max_j \sum_{i=1}^n |g_{ij}(t_0, t_f; z)|,$$

where $G(t_0, t_f; z)^{-1} = \{g_{ij}(t_0, t_f; z)\}$. Let Ω be the image of ψ:

$$\Omega \equiv \{x | x = P(z), z \in \psi\}. \tag{17}$$

It is clear that the operator P as defined by (11) is continuous and it is easily established from the Arzela–Ascoli theorem [6] that the image set Ω defined by (17) is compact and is a subset of ψ defined by (14). Hence by Schauder's theorem [6], the operator has a fixed point.

Proof of Theorem 1. The significance of Lemma 3 is that there always exists at least one function $z^* \in C_n[t_0, t_f]$, which, introduced into (10), provides an x^* such that $x^* = z^*$. This x^*, however, is a solution to system (1) for the control input $u(t_0, t, t_f; z^*)$, which is easily verified by differentiating x^* with respect to t. Since $u(t_0, t, t_f; z^*)$ takes system (1) from x_0 to x_f on the interval $[t_0, t_f]$, and since by Lemma 3 there is a $u(t_0, t, t_f; z^*)$ for all $x_0, x_f \in R^n$, system (1) is *globally controllable.* In particular, if condition (C(a)) of Theorem 1 holds, the above conclusion is true for *some* finite time interval $[t_0, t_f]$, and the system is *completely controllable.* If condition (C(b)) of Theorem 1 holds, the above conclusion is true for *every* finite time interval $[t_0, t_f]$, and the system is therefore *totally controllable.*

4. Local controllability. The method used to establish Theorem 1 for global controllability can be used to derive a theorem for local controllability under less restrictive conditions; i.e., it will no longer be necessary that the elements of A and B be bounded for all $x \in R^n$, and the Gramian determinant need only have a lower bound on a bounded set of functions z. This bounded set is defined by

$$\mathscr{C} = \{z | z = C_n[t_0, t_f]; z(t_0) = x_0, z(t_f) = x_f; x_0, x_f \in R^n; \|z\| \leqq K_1\}, \tag{18}$$

where K_1 is some real positive nonzero constant.

THEOREM 2 (Local controllability). *The system*

$$dx/dt = A(t, x)x + B(t, x)u$$

is locally, (a) *completely state controllable at* t_0, *or* (b) *totally state controllable, about the origin if the following three conditions all hold:*

(A) *The elements* $a_{ik}(t, x)$ *of* A $(i, k = 1, 2, \cdots, n)$ *and* $b_{il}(t, x)$ *of* B $(l = 1, 2, \cdots, m,\ i = 1, 2, \cdots, n)$ *are piecewise continuous functions of* t *and continuous functions of* x.

(B) $|a_{ik}(t, z)| \leqq M, |b_{il}(t, z)| \leqq N$ *for all* $z \in \mathscr{C}$, $t \in [t_0, t_f]$, *where* M *and* N *are positive real constants.*

(C) $$\inf_{z \in \mathscr{C}} \det G(t_0, t_f; z) \geqq c \quad \textit{for some } c > 0$$

(a) *for some* $t_f > t_0$, *in the case of complete state controllability at* t_0,

(b) *for all* t_0 *and for all* $t_f > t_0$, *in the case of total state controllability.*

The proof of the theorem is based on the following lemma, which is a counterpart to Lemma 3.

LEMMA 4. *The operator* P *defined by* (11) *has a fixed point in* $\mathscr{C}$ *defined by* (18) *such that* $x^* = P(x^*)$, *if* $|x_0| < K_2$, $|x_f| < K_3$, *where* K_2 *and* K_3 *are real positive constants which are sufficiently small and not both zero, and if conditions* (A) *to* (C) *of Theorem* 2 *are fulfilled.*

Proof of Lemma 4. The proof follows exactly the same reasoning as Lemma 3.

Proof of Theorem 2. The proof is the same as for Theorem 1, if one uses $\mathscr{C}$ instead of $C_n[t_0, t_f]$ and Lemma 4 instead of Lemma 3.

5. Relation of Gramian matrix to controllability matrix. A serious difficulty in the application of Theorems 1 or 2 is to show that condition (C) holds. Therefore a relation which shows that condition (C) holds (at least for certain cases) will now be given.

If the additional assumption is introduced that $A(t, z)$ and $B(t, z)$ are piecewise differentiable on $[t_0, t_f]$ at least $n - 2$ and $n - 1$ times, respectively, then the controllability matrix Q of Silverman and Meadows [5] can be introduced.

Define the matrix

$$(19)\quad Q(t; z, z^{(1)}, \cdots, z^{(n-1)}) = [P_0(t; z), P_1(t; z, z^{(1)}), \cdots, P_{n-1}(t; z, z^{(1)}, \cdots, z^{(n-1)})],$$

where $P_k(t; z, z^{(1)}, \cdots, z^{(k)})$ is recursively defined by

$$(20)\quad \begin{aligned} P_k(t; z, z^{(1)}, \cdots, z^{(k)}) = &-A(t, z)P_{k-1}(t; z, z^{(1)}, \cdots, z^{(k-1)}) \\ &+ \frac{d}{dt}P_{k-1}(t; z, z^{(1)}, \cdots, z^{(k-1)}), \end{aligned}$$

$$(21)\quad P_0(t, z) = B(t, z).$$

For simplicity denote $Q(t; z, z^{(1)}, \cdots, z^{(n-1)})$ by $Q(t, z)$. The results obtained in [7] allow the formulation of the following theorem.

THEOREM 3. *Assume that* $A(t, z)$ *and* $B(t, z)$ *of* (3) *are piecewise differentiable on* $[t_0, t_f]$ *at least* $n - 2$ *and* $n - 1$ *times, respectively, and that* $B(t, z)$ *is an* $n \times 1$ *vector. If* $\inf_{z \in C_n[t_0, t_f]\ (\text{or } \mathscr{C})} [\det Q(t, z)]^2 \geqq \gamma$ *for some* $\gamma > 0$ *and for some* t *in* $[t_\alpha, t_\beta]$, *where* $[t_\alpha, t_\beta]$ *is a subinterval of* $[t_0, t_f]$, *then* $\det G(t_\alpha, t_\beta; z)$ *of Theorems* 1 *and* 2 *has a lower bound such that*

$$\inf_{z \in C_n[t_0, t_f]\,(or\,\mathscr{C})} \det G(t_\alpha, t_\beta; z) \geqq \varepsilon \quad \textit{for some } \varepsilon > 0.$$

6. Some numerical examples.

Example 1. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 + \sin[g(x_1, x_2, t)]u, \\ \dot{x}_2 &= -x_1 + \sin[g(x_1, x_2, t)]u, \end{aligned} \tag{22}$$

where $g(x_1, x_2, t)$ is a continuous function of x_1, x_2 and a piecewise continuous function of t and satisfies the following inequality:

$$0 < \varepsilon \leqq g(x_1, x_2, t) \leqq \pi - \varepsilon \qquad \text{for all } x_1, x_2 \in C_n[t_0, t_f], \quad t \in [t_0, t_f].$$

By using Theorems 1 and 3 global total controllability is easily established. The coefficients of x_1, x_2 and u fulfill the conditions (A) and (B) of Theorem 1. Condition (C(b)) is established by using Theorem 3. The determinant of the controllability matrix is

$$\det Q(t, z) = \begin{vmatrix} \sin[g(z_1, z_2, t)] & \sin[g(z_1, z_2, t)] - \dfrac{d}{dt}\sin[g(z_1, z_2, t)] \\ \sin[g(z_1, z_2, t)] & -\sin[g(z_1, z_2, t)] - \dfrac{d}{dt}\sin[g(z_1, z_2, t)] \end{vmatrix} \tag{23}$$

which yields

$$\det Q(t, z) = -2\sin^2[g(z_1, z_2, t)]. \tag{24}$$

From (24) the smallest lower bound is readily determined:

$$\inf_{z \in C_n[t_0, t_f]} [\det Q(t, z)]^2 \geqq 4 \sin^4 \varepsilon > 0, \tag{25}$$

which holds for all t. Hence by Theorem 3, condition (C(b)) of Theorem 1 is satisfied, and system (22) is therefore globally totally controllable.

Example 2. Consider the system

$$\dot{x} = \begin{pmatrix} 0 & \dfrac{1}{1 - x_1^2 x_2^2} \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u. \tag{26}$$

For $|\max(x_1, x_2)| \leqq K < 1$, conditions (A) and (B) of Theorem 2 are satisfied. Theorem 3 will be used to establish the third condition. The determinant of the controllability matrix is

$$\det Q(t, z) = \frac{-1}{1 - z_1^2 z_2^2}, \tag{27}$$

and

$$\inf_{z \in \mathscr{C}} [\det Q(t, z)]^2 = \inf_{z \in \mathscr{C}} \left(\frac{-1}{1 - z_1^2 z_2^2}\right)^2 = \frac{1}{(1 - K^4)^2}. \tag{28}$$

Hence condition (C(b)) of Theorem 2 holds and system (26) is locally totally controllable about the origin.

7. A class of nonlinear systems which is globally controllable. Theorems 1 and 3 can be used to establish classes of nonlinear systems which are globally controllable. The following is an example.

THEOREM 4. *Consider the system*

$$\dot{x} = A(x, t)x + B(x, t)u, \tag{29a}$$

where $A(x, t)$ and $B(x, t)$ have the following form:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & 0 & \cdots & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & 0 & & 0 \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ & & & & a_{n-2,n-1} & & 0 \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} & & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-1} & & a_{n,n} \end{bmatrix}, \tag{29b}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ b_n \end{bmatrix}; \tag{29c}$$

then a sufficient condition that system (29) *be*

(a) *globally*

  (i) *completely state controllable at t_0,*

  (ii) *totally state controllable,*

(b) *locally about the origin*

  (i) *completely state controllable at t_0,*

  (ii) *totally state controllable*

is that the following three conditions all hold:

(A) *The elements $a_{ik}(t, z)$ of $A(i, k = 1, 2, \cdots, n)$ are piecewise differentiable on $[t_0, t_f]$ at least $n - 2$ times and $b_n(t, z)$ is piecewise differentiable on $[t_0, t_f]$ at least $n - 1$ times.*

(B) (a) *In the case of global controllability*

$$|a_{il}(t, x)| \leqq M, \qquad |b_n(t, x)| \leqq N \qquad \text{for all } x \in R^n, \quad t \in [t_0, t_f].$$

(b) *In the case of local controllability*

$$|a_{il}(t, z)| \leqq M, \qquad |b_n(t, z)| \leqq N \qquad \text{for all } z \in \mathscr{C}, \quad t \in [t_0, t_f],$$

where $\mathscr{C}$ *is defined by* (18).

(C) (a) *In the case of global controllability, there exists a constant* $c > 0$ *such that*

$$b_n^2(t_j, z) \geqq c, \qquad a_{i,i+1}^2(t_j, z) \geqq c \qquad \text{for all } z \in C_n[t_0, t_f],$$

$i = 1, 2, \cdots, n-1$, *for some* $t_j \in [t_0, t_f]$.

(b) *In the case of local controllability, there exists a constant* $c > 0$ *such that*

$b_n^2(t_j, z) \geqq c$, $a_{i,i+1}^2(t_j, z) \geqq c$ *for all* $z \in \mathscr{C}$, $i = 1, 2, \cdots, n-1$, *for some* $t_j \in [t_0, t_f]$,

(i) *for some* $t_f > t_0$ *in the case of complete state controllability at* t_0,

(ii) *for all* t_0 *and for all* $t_f > t_0$, *in the case of total state controllability.*

Proof. It is easily established for (29), that

$$\det Q(t, z) = b_n^n \prod_{i=1}^{n-1} a_{i,i+1}^i. \tag{30}$$

Theorem 4 immediately follows on using this result together with Theorems 1, 2 and 3.

Remark. It is seen then that for the class of systems given by (29), the system is globally totally controllable if all the elements are bounded and if the product of the superdiagonal elements of A with b_n is equal to zero at most only a countable number of times. This is a generalization of the nonlinear time-varying system considered in [8] which is as follows:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ & & & & 0 & 1 \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-1} & a_{n,n} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u. \tag{31}$$

Acknowledgment. Initial discussion of the problem with Professors L. M. Silverman and P. P. Varaiya is greatly appreciated.

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